

# On existence of semi-wavefronts for a non-local reaction-diffusion equations with distributed time delay

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## Abstract

We establish the existence of semi-wavefronts solutions for a non-local delayed reaction-diffusion equation with monostable nonlinearity. The existence result is proved for all speeds  $c \geq c_*$ , where the determination of  $c_*$  is similar to the calculation of the minimal speed of propagation. The results are applied to some non-local reaction-diffusion epidemic and population models with distributed time delay.

*Keywords:* time-delayed reaction-diffusion equation; positive wavefront; non-local interaction; minimal wave; semi-wavefront; existence.

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## 1. Introduction.

The main object of study in this paper is the non-local reaction-diffusion equation

$$u_t(t, x) = u_{xx}(t, x) - f(u(t, x)) + \int_0^\infty \int_{\mathbb{R}} K(s, w) g(u(t-s, x-w)) dw ds, \quad (1.1)$$

where the time  $t \geq 0$ ,  $x \in \mathbb{R}$ , it is assumed that the non-negative averaging kernel  $K$  satisfies the typical condition

$H_0$ :  $K \in L^1(\mathbb{R}_+ \times \mathbb{R})$  and  $\int_0^\infty \int_{\mathbb{R}} K(s, w) dw ds = 1$ . Moreover, for any  $c \in \mathbb{R}$ , there exists some  $\gamma^\# := \gamma^\#(c) \in (0, +\infty]$  such that  $\int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} dw ds < \infty$  for each  $z \in [0, \gamma^\#)$  and diverges, if  $z > \gamma^\#$ .

Equation (1.1), with appropriate  $f, g$  and  $K$ , is often used to model ecological and biological processes where the typical interpretation of  $u(t, x)$  is the population density of mature species. The last investigations shows that asymmetric kernels can be present in the growth dynamics of single-species population (see, e.g. [4, 5, 6, 9, 10, 11, 12, 13, 16, 17, 18, 19]). For example, the type of asymmetry can occur in the population marine models when the population juveniles move by advection as well as diffusion, but the population adults move by diffusion alone (see [9, Applications]). In this way, we are interested in equation (1.1) when  $K$  is asymmetric.

We also assume the following conditions on the monostable nonlinearity  $g$  and the function  $f$ :

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$H_1$ :  $g \in C(\mathbb{R}_+, \mathbb{R}_+)$  is such that  $g(0) = 0$ ,  $g(s) > 0$  for all  $s > 0$ , and differentiable at 0.

$H_2$ :  $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$  is strictly increasing with  $f(0) = 0$ ,  $0 < f'(0) < g'(0)$  and  $f(+\infty) > \sup_{s \geq 0} g(s)$ .

In this work we will study the existence of semi-wavefront solutions of equation (1.1), i.e. bounded positive continuous non-constant waves  $u(t, x) = \phi(x + ct)$ , propagating with speed  $c$ , and satisfying the boundary condition  $\phi(-\infty) = 0$ . An important special case of semi-wavefront is a wavefront, i.e. positive classical solution  $u(t, x) = \phi(x + ct)$  satisfying  $\phi(-\infty) = 0$  and  $\phi(+\infty) = \kappa$ . During the last time, the existence of semi-wavefront or wavefront solutions for equation (1.1) have been investigated in several papers assuming different conditions on  $f$ ,  $K$  and  $g$  (see e.g. (see, [2, 7, 8, 9, 14, 15, 20, 21, 22, 25, 27])). However, a few works have considered the existence problem for the general non-local reaction-diffusion equation with distributed delay (1.1) (see [5, 19, 25]). In any case, for the non-local reaction-diffusion equation with distributed delay (1.1), the existence problem of the semi-wavefront is still unsolved in the general case when  $K$  is asymmetric. In this paper we apply the techniques of [9] to present a solution of this open problem, weakening or removing some typical restrictions on nonlinearities.

Now, we present our main results:

**Theorem 1.1.** *Assume that  $H_0$ - $H_2$  hold. If there exists  $L \geq g'(0)$  such  $g(s) \leq Ls$  for all  $s \geq 0$ , then there exists  $c_* \in \mathbb{R}$  such that the equation (1.1) has a semi-wavefront solution  $u(x, t) = \phi(x + ct)$  propagating with speed  $c \geq c_*$ . Furthermore, if equation  $f(s) = g(s)$  has only two solutions: 0 and  $\kappa$ , with  $\kappa$  being globally attracting with respect to  $f^{-1} \circ g$ , then the equation (2.1) has at least one wavefront  $u(x, t) = \phi(x + ct)$  propagating with speed  $c \geq c_*$  such that  $\phi(+\infty) = \kappa$ .*

*Remark 1.2.* We observe that Theorem 1.1 shows that the condition

$$g(s) \leq Ls, \quad s \geq 0, \quad (1.2)$$

when  $L = g'(0)$  is not at all obligatory to prove the existence of fast semi-wavefronts solution. Moreover, our result also incorporates asymmetric kernels and the critical case. Thus Theorem 1.1 improves or completes the existence results in [5, 19, 25], where the existence was established for  $g$  satisfying (1.2) and assuming either even or Gaussian kernel  $K$ .

The paper is organized as follows. Section 2 contains some preliminary results. In Section 3, we show some geometric properties of the bounded solutions. In the fourth section, we our main results are proved. In the last section some applications are presented.

## 2. Preliminaries.

This section contains some preliminary results and transformations are needed to apply the methods of [9].

First, note that the profile  $y = \phi$  of the semi-wavefront solution  $u(t, x) = \phi(x + ct)$  to (1.1) must satisfy the equation

$$y''(t) - cy'(t) - f(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s, w)g(y(t - cs - w))dw ds = 0 \quad (2.1)$$

for all  $t \in \mathbb{R}$ . Note that this equation can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s, w)g(y(t - cs - w))dw ds = 0, \quad (2.2)$$

where  $f_\beta(s) = \beta s - f(s)$  and  $\beta$  is chosen large enough such that  $\beta > f'(0)$ . Thus in order to establish the existence of semi-wavefront solution to (1.1), we have to prove the existence of positive bounded solution  $\phi$  of equation (2.1), satisfying  $\phi(-\infty) = 0$ .

Now, being  $\phi$  a positive bounded solution to (2.1), it should satisfy the integral equation

$$\begin{aligned} \phi(t) &= \frac{1}{\sigma(c)} \left( \int_{-\infty}^t e^{\nu(c)(t-s)} (\mathcal{G}\phi)(s) ds + \int_t^{+\infty} e^{\mu(c)(t-s)} (\mathcal{G}\phi)(s) ds \right) \\ &= \int_{\mathbb{R}} k_1(t-s) (\mathcal{G}\phi)(s) ds, \quad t \in \mathbb{R}, \end{aligned} \quad (2.3)$$

where

$$k_1(s) = (\sigma(c))^{-1} \begin{cases} e^{\nu(c)s}, & s \geq 0 \\ e^{\mu(c)s}, & s < 0 \end{cases},$$

$\sigma(c) = \sqrt{c^2 + 4\beta}$ ,  $\nu(c) < 0 < \mu(c)$  are the roots of  $z^2 - cz - \beta = 0$  and the operator  $\mathcal{G}$  is defined as

$$(\mathcal{G}\phi)(t) := \int_0^\infty \int_{\mathbb{R}} K(s, w)g(\phi(t - cs - w))dw ds + f_\beta(\phi(t)).$$

Note that  $(\mathcal{G}\phi)(t)$  can be rewritten as

$$\begin{aligned} (\mathcal{G}\phi)(t) &= \int_{\mathbb{R}} g(\phi(t-r)) \int_0^\infty K(s, r - cs) ds dr + f_\beta(\phi(t)) \\ &= \int_{\mathbb{R}} g(\phi(t-r)) k_2(r) dr + f_\beta(\phi(t)), \end{aligned} \quad (2.4)$$

where, by Fubini's Theorem,

$$k_2(r) = \int_0^\infty K(s, r - cs) ds,$$

is well defined for all  $r \in \mathbb{R}$ . Finally, from (2.4) we get that  $\phi$  also must satisfy the equation

$$\phi(t) = (k_1 * k_2) * g(\phi)(t) + k_1 * f_\beta(\phi)(t), \quad t \in \mathbb{R}, \quad (2.5)$$

where  $*$  denotes convolution  $(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds$ .

In order to apply some results of [9], we rewrite equation (2.5) as

$$\phi(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) g(\phi(t-s), \tau) ds, \quad t \in \mathbb{R}, \quad (2.6)$$

where

$$\mathcal{N}(s, \tau) = \begin{cases} (k_1 * k_2)(s), & \tau = \tau_1, \\ k_1(s), & \tau = \tau_2, \end{cases} \quad g(s, \tau) = \begin{cases} g(s), & \tau = \tau_1, \\ f_\beta(s), & \tau = \tau_2, \end{cases}$$

and  $X = \{\tau_1, \tau_2\}$ . Thus we can invoke the theory developed in [9] to prove the existence of positive bounded solution of (2.1), vanishing at  $-\infty$ .

Now we have to introduce several definitions. Let  $c_*$  [respectively,  $c_*$ ] be the minimal value of  $c$  for which

$$\chi_0(z, c) := z^2 - cz - f'(0) + g'(0) \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} dw ds,$$

[respectively,

$$\chi_L(z, c) := z^2 - cz - \inf_{s \geq 0} f'(s) + L \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} dw ds, \quad L \geq g'(0)]$$

has at least one positive root. We observe that  $c_* \geq c_*$  and the function  $\chi_0(z, c)$  is associated with the linearization of (2.1) along the trivial equilibrium. Moreover, we also introduce the characteristic function  $\chi$  associated with the variational equation along the trivial steady state of (2.6), by

$$\chi(z) := 1 - \int_{\mathbb{R}} \int_X \mathcal{N}(s, \tau) g'(0, \tau) d\rho(\tau) e^{-sz} ds.$$

Observe that

$$\begin{aligned} \chi(z) &= 1 - g'(0) \int_{\mathbb{R}} \mathcal{N}(s, \tau_1) e^{-zs} ds - (\beta - f'(0)) \int_{\mathbb{R}} \mathcal{N}(s, \tau_2) e^{-zs} ds \\ &= 1 - \frac{\beta - f'(0)}{\beta + cz - z^2} - \frac{g'(0)}{\beta + cz - z^2} \int_0^\infty \int_{\mathbb{R}} K(r, w) e^{-z(rc+w)} dw dr \\ &= -\frac{\chi_0(z, c)}{\beta + cz - z^2}. \end{aligned} \quad (2.7)$$

Note that the zeros of function  $\chi(z)$  are determined by the roots of characteristic equation  $\chi_0(z, c) = 0$  and

$$\chi(0) = -\frac{\chi_0(0, c)}{\beta} = \frac{f'(0) - g'(0)}{\beta} < 0. \quad (2.8)$$

We also will need the following function

$$\chi_L(z) := 1 - \int_{\mathbb{R}} \int_X \mathcal{N}(s, \tau) C(\tau) d\rho(\tau) e^{-sz} ds,$$

where

$$C(\tau) = \begin{cases} L, & \tau = \tau_1, \\ \beta - \inf_{s \geq 0} f'(s), & \tau = \tau_2, \end{cases} \quad (2.9)$$

is measurable function on  $(X, \mu)$  with  $L \geq g'(0)$ . Similarly, note that

$$\chi_L(z) = -\frac{\chi_L(z, c)}{\beta + cz - z^2} \quad (2.10)$$

and

$$\chi_L(0) = \frac{\inf_{s \geq 0} f'(s) - L}{\beta} < 0.$$

The properties of the real solutions of the equations  $\chi_0(z, c) = \chi_L(z, c) = 0$  are shown in the following lemma. We are considering a more general equation:

$$\mathcal{R}(z, c) := z^2 - cz - q + p \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} dw ds = 0,$$

where  $p > q > 0$ .

**Lemma 2.1.** *Suppose that given  $c \in \mathbb{R}$ , the function  $\mathcal{R}(z, c)$  is defined for all  $z$  from some maximal interval  $[0, \delta(c))$ ,  $\delta(c) \in (0, +\infty]$ . Then there exists  $c^\# \in \mathbb{R}$  such that*

- (i) *for any  $c > c^\#$ , the function  $\mathcal{R}(z, c)$  has at least one positive zero  $z = \lambda_1(c) \in (0, \delta(c))$ , it may have at most two positive zeros on  $(0, \delta(c))$  and it does not have any negative zero. If the second zero exists, we denote it as  $\lambda_2(c) > \lambda_1(c)$ . Furthermore, each  $\lambda_j(c) < \mu_q(c)$ , where  $\mu_q(c) > 0$  satisfies the equation  $z^2 - cz - q = 0$ .*
- (ii) *if  $c = c^\#$  and  $\lim_{z \uparrow \delta(c^\#)} \mathcal{R}(z, c^\#) \neq 0$ , then  $\mathcal{R}(z, c^\#)$  has a unique double root on  $(0, \delta(c^\#))$ , denoted by  $z = \lambda_1(c^\#)$ , and  $\mathcal{R}(z, c^\#) > 0$  for all  $z \neq \lambda_1(c^\#) \in [0, \delta(c^\#))$ .*

*Proof.* See [1, Lemma 3.1]. □

**Example 2.2.** We consider a space structured population with maturation effects described by the delays, for example marine species, where the juveniles move by advection as well as diffusion, but the adults move by diffusion alone. If  $u(t, x)$  denote the density of the population adult in the location  $x \in \mathbb{R}$  and time  $t$ , then  $u(t, x)$  can be represented by the following model:

$$u_t(t, x) = d_a u_{xx}(t, x) - \mu_a u(t, x) + \int_0^\infty \int_{\mathbb{R}} g(u(t-s, x-w)) \frac{\mu_j e^{\frac{-(w+v_j s)^2}{4d_j s} - \mu_j s}}{2\sqrt{\pi d_j s}} dw ds, \quad (2.11)$$

where  $g$  is the birth function,  $d_j, v_j, \mu_j$  are respectively the diffusion rate, the advection velocity and the death rate for juveniles and  $d_a, \mu_a$  are respectively the diffusion rate and the death rate for adults population (see [9] for more details). Note that spatial asymmetry occurs in this model with

$$K(s, w) = \frac{\mu_j e^{\frac{-(w+v_j s)^2}{4d_j s} - \mu_j s}}{2\sqrt{\pi d_j s}}.$$

By scaling of variables, we can suppose that  $d_a = 1$ . Thus the characteristic function  $\chi(z, c)$  associated with the linearization of equation (2.11) along the trivial steady state is given by the following form

$$\chi(z, c) := z^2 - cz - \mu_a + p \int_0^\infty \int_{\mathbb{R}} \frac{\mu_j e^{\frac{-(w+v_j s)^2}{4d_j s} - \mu_j s}}{2\sqrt{\pi d_j s}} e^{-z(cs+w)} dw ds = 0,$$

where  $p := g'(0) > \mu_a$ . A simple calculation of the integral allows to obtain that

$$\chi(z, c) := z^2 - cz - \mu_a + \frac{p\mu_j}{\mu_j + (c - v_j)z - d_j z^2}.$$

Note that  $\chi(0, c) = p - \mu_a > 0$  and  $\lim_{c \downarrow -\infty} \chi(z, c) = +\infty$  for  $z \in (0, +\infty)$ . In addition,

$$\frac{\partial^2 \chi}{\partial z^2}(z, c) > 0, \quad z \in [0, +\infty),$$

the function  $\chi(z, c)$  is strictly convex with respect to  $z$ , and hence it has at most two real zeros for each  $c$ . For example, in the particular case when the advection for juveniles is  $v_j = 0.02$ , diffusivity  $d_j = 100$  and  $\mu_j = 0.001, \mu_a = 0.05$  are the death rates for juveniles and for adults population, respectively, figures 1 and 2 show the behavior of  $\chi(z, c)$  when  $z \geq 0$  and  $p = 2$ .

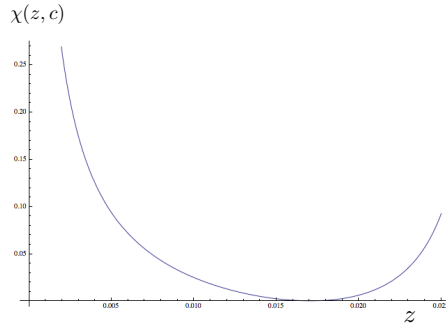


Figure 1:  $\chi(z, c), z \geq 0$  for  $c = 2.854$ .

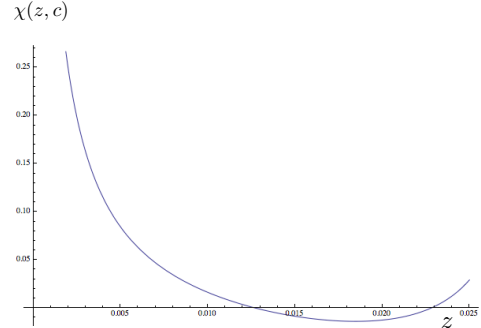


Figure 2:  $\chi(z, c), z \geq 0$  for  $c = 3$ .

### 3. Some geometric properties

Now we show some geometric properties of the bounded solutions to equation (2.1).

**Lemma 3.1.** *If  $u(t, x) = \phi(x + ct) \geq 0$  is a bounded solution of equation (2.1) such that  $\phi$  vanishes at some point, then  $\phi \equiv 0$ .*

*Proof.* Suppose that there exists  $t_0 \in \mathbb{R}$  such that  $\phi(t_0) = 0$ . From (2.3) we get that

$$\phi(t_0) = \int_{\mathbb{R}} k_1(t_0 - s)(\mathcal{G}\phi)(s)ds = 0.$$

Since  $k_1(t) > 0$  and  $(\mathcal{G}\phi)(t) \geq 0$  for all  $t \in \mathbb{R}$ , we necessarily have  $(\mathcal{G}\phi)(t) = 0$  for all  $t \in \mathbb{R}$ .

In this way, according to  $f(0) = g(0) = 0$  and since  $f$  is continuous on  $[0, \sup_{t \in \mathbb{R}} \phi(t)]$ , we can choose  $\beta > 0$  sufficiently large such that  $f_\beta(s) = \beta s - f(s) \geq 0$  for all  $s \in (0, \sup_{t \in \mathbb{R}} \phi(t)]$ . Thus we have

$$\int_{\mathbb{R}} g(\phi(t - r))k_2(r)dr = f_\beta(\phi(t)) = 0, \quad t \in \mathbb{R},$$

which implies that  $\phi(t) = 0$  for all  $t \in \mathbb{R}$ . □

**Lemma 3.2.** Assume that  $H_0$ - $H_2$  and (1.2) hold. Let  $u(t, x) = \phi(x + ct) \geq 0$  be a bounded solution of equation (2.1) with speed  $c \geq c_*$ . If  $\phi(-\infty) = 0$ , then  $\liminf_{t \rightarrow +\infty} \phi(t) \geq \delta(\phi) > 0$  for some  $\delta(\phi) > 0$ .

*Proof.* First, we observe that  $\chi(0) < 0$ , by (2.8). In addition, the conditions (1.2) imply that  $g(s, \tau_1) \leq Ls$  for all  $s \geq 0$ . In this way, from [1, Lemma 4.1] we get that for  $\delta > 0$ , there exists  $\beta = \beta(\delta) > 0$  sufficiently large such that  $f_\beta(s) \geq 0$  for all  $s \geq 0$  and

$$f_\beta(s) \leq \left( \beta - \inf_{s \geq 0} f'(s) \right) s, \quad s \in [0, \delta].$$

Consequently,  $g(s, \tau_2) \leq \left( \beta - \inf_{s \geq 0} f'(s) \right) s$  and  $g(s, \tau) \leq C_\delta(\tau)s$  for all  $s \in [0, \delta]$ , where  $C_\delta(\tau_1) = L$  and  $C_\delta(\tau_2) = \beta - \inf_{s \geq 0} f'(s)$ . Note that  $C_\delta(\tau) \geq 0$  is a measurable function and

$$\begin{aligned} \int_X C_\delta(\tau) d\mu(\tau) \int \mathcal{N}(s, \tau) ds &= L \int_{\mathbb{R}} (k_1 * k_2)(s) ds + \left( \beta - \inf_{s \geq 0} f'(s) \right) \int_{\mathbb{R}} k_1(s) ds \\ &= 1 + \frac{L - \inf_{s \geq 0} f'(s)}{\beta} < +\infty. \end{aligned}$$

Finally, from Lemma 3.1 we obtain that all the hypotheses of [9, Theorem 3] hold. In consequence, we can conclude that either  $\phi(+\infty) = 0$  or  $\liminf_{t \rightarrow +\infty} \phi(t) > 0$ . Since  $\phi(-\infty) = 0$  and all real zeros of  $\chi(z) = 0$  are positive for each  $c \geq c_*$ , [9, Corollary 4] implies that  $\liminf_{t \rightarrow +\infty} \phi(t) \geq \delta(\phi) > 0$  for some  $\delta(\phi) > 0$ . □

**Lemma 3.3.** Suppose that conditions (1.2) and  $H_0$ - $H_2$  hold. Furthermore, we also assume that

$$f(s) \geq \inf_{s \geq 0} f'(s)s, \quad s \geq 0. \quad (3.1)$$

Let  $\phi$  be a positive solution of (2.1) with speed  $c \geq c_*$  and  $\lambda$  be a positive root of equation  $\chi_L(z, c) = 0$ . If the solution  $\phi$  satisfies the inequality  $\phi(t) \leq \delta e^{\lambda t}$  for all  $t \in \mathbb{R}$  and for some  $\delta > 0$ , then  $\phi$  is bounded on  $\mathbb{R}$  and

$$\phi(t) \leq \frac{\sup_{t \geq 0} g(t)}{\inf_{t \geq 0} f'(t)}.$$

*Proof.* First, it is clear that  $f_\beta(t) \leq (\beta - \inf_{s \geq 0} f'(s))t$  for all  $t \geq 0$ . Thus, being  $\phi$  a positive solution of (2.6), we obtain that

$$\begin{aligned}\phi(t) &\leq \sup_{u \geq 0} g(u) \int_{\mathbb{R}} (k_1 * k_2)(s) ds + \delta(\beta - \inf_{s \geq 0} f'(s)) \int_{\mathbb{R}} k_1(s) e^{\lambda(t-s)} ds \\ &= \frac{\sup_{u \geq 0} g(u)}{\beta} + \delta(\beta - \inf_{s \geq 0} f'(s)) e^{\lambda t} \int_{\mathbb{R}} k_1(s) e^{-\lambda s} ds.\end{aligned}$$

In consequence, by definition of function  $k_1$ , we get that

$$\phi(t) \leq \frac{\sup_{u \geq 0} g(u)}{\beta} + \frac{\delta(\beta - \inf_{s \geq 0} f'(s))}{\beta + c\lambda - \lambda^2} e^{\lambda t}. \quad (3.2)$$

Now, using the inequality (3.2) and applying this argument again we obtain by induction that

$$\phi(t) \leq \delta e^{\lambda t} \theta^n + \frac{\rho(1 - \gamma^{n+1})}{1 - \gamma}, \quad n \in \mathbb{N},$$

where  $\rho := \frac{\sup_{u \geq 0} g(u)}{\beta}$ ,  $\theta := \frac{\beta - \inf_{s \geq 0} f'(s)}{\beta + c\lambda - \lambda^2}$  and  $\gamma := \frac{\beta - \inf_{s \geq 0} f'(s)}{\beta}$ . It is clear that  $\gamma < 1$ . Moreover, since  $\lambda$  is a positive root of equation  $\chi_L(z, c) = 0$ , it is easy to check that also  $\theta < 1$ . Thus, by passing to the limit as  $n \rightarrow \infty$ , we obtain the estimate

$$\phi(t) \leq \frac{\rho}{1 - \gamma} = \frac{\sup_{u \geq 0} g(u)}{\inf_{s \geq 0} f'(s)}.$$

□

We now establish some properties of  $\mathcal{N}(s, \tau)$  and  $g(s, \tau)$ , which will be necessary to apply methods of [9] in order to obtain the uniform persistence of the positive solutions to equation (2.1).

**Lemma 3.4.** *Assume that  $H_0$ - $H_2$  hold. Furthermore, suppose that the derivative  $f'$  is locally bounded. Then the following statements are valid:*

- (i) *The function  $\tilde{g}(v) := \int_{\mathbb{R}} \int_{X \setminus \{\tau_1\}} g(v, \tau) \mathcal{N}(s, \tau) d\rho(\tau) ds$  is a monotone increasing function.*
- (ii) *There exists  $\xi_2 > 0$  such that the function  $\theta(v) := v - \tilde{g}(v)$  is strictly increasing on  $[0, +\infty)$ , and  $\theta(\xi_2) > \sup_{v \geq 0} g(v, \tau_1) \int_{\mathbb{R}} \mathcal{N}(s, \tau_1) ds$ .*
- (iii) *Define  $G(v) := \theta^{-1}(\frac{1}{\beta} g(v, \tau_1))$ . Then  $G(0) = 0$ ,  $0 < G(v) < \xi_2$ ,  $v > 0$ . Furthermore,  $G'(0)$  is finite and  $G'(0) > 1$ .*

*Proof.* First, note that

$$\tilde{g}(v) = \int_{\mathbb{R}} g(v, \tau_2) \mathcal{N}(s, \tau_2) d\rho(\tau) ds = f_\beta(v) \int_{\mathbb{R}} k_1(s) ds = \frac{f_\beta(v)}{\beta}.$$

Since  $f$  is strictly increasing and  $f'$  is locally bounded, the function  $\tilde{g}(v) = \frac{\beta v - f(v)}{\beta}$  is monotone increasing on  $\mathbb{R}$  for some  $\beta$  sufficiently large. Moreover,  $\theta(v) = v - \frac{f_\beta(v)}{\beta} =$



$\frac{f(v)}{\beta}$ ,  $v \geq 0$ , is also strictly increasing on  $[0, +\infty)$ . Now, consider  $\xi_2 > 0$  such that  $f(\xi_2) > \sup_{s \geq 0} g(s)$ . Then

$$\theta(\xi_2) = \frac{f(\xi_2)}{\beta} > \frac{1}{\beta} \sup_{v \geq 0} g(v) = \sup_{v \geq 0} g(v) \int_{\mathbb{R}} k_1(s) ds = \sup_{v \geq 0} g(v, \tau_1) \int_{\mathbb{R}} \mathcal{N}(s, \tau_1) ds.$$

On the other hand, an easy computation shows that  $G(v) = \theta^{-1}(\frac{1}{\beta}g(v, \tau_1)) = f^{-1}(g(v))$  and  $G(0) = 0$ . In addition, if  $y = \theta^{-1}(\frac{1}{\beta}g(v_0)) > \xi_2$  for some  $v_0 > 0$ , then we have  $f(\xi_2) < g(v_0)$ , a contradiction. Hence  $G(v) < \xi_2$ .

Finally, since  $g'(0) > f'(0) > 0$ , it is clear that  $1 < G'(0) = \frac{g'(0)}{f'(0)} < \infty$ , by Inverse Function Theorem.  $\square$

*Remark 3.5.* We observe that the function  $G$  has several other important properties which are also necessary to prove the uniform persistence of the positive solution to (2.1), see [9, Lemma 5].

The next lemma establishes the uniform persistence property of semi-wavefront to (2.1) holds.

**Lemma 3.6.** *Assume all hypotheses of Lemma 3.4 are fulfilled and suppose the condition (1.2). Let  $u = \phi(x + ct)$  be a positive bounded solution of equation (2.1) with speed  $c \geq c_*$  and  $\xi_1 \in (0, \xi_2)$ , where  $\xi_2$  is as in Lemma 3.4, such that  $\xi_1 > \inf_{t \in \mathbb{R}} \phi(t)$ . Then  $\phi(-\infty) = 0$  and  $\liminf_{t \rightarrow +\infty} \phi(t) > \xi_1$ .*

*Proof.* The proof follows from [9, Theorem 6]. In fact, we need the conditions **(C)**, **(P)** and **(N)** of [9] to be satisfied. In this way, it is clear that the hypothesis **(C)** holds and  $\chi(0) < 0$ . From Lemma 3.4 we obtain that the hypothesis **(N)** also holds and the condition **(P)** is obtained from Lemma 3.1.  $\square$

#### 4. The existence

Throughout all this section, we assume conditions (1.2) and (3.1) hold, and that the speed  $c \geq c_*$ . Let  $\lambda > 0$  be the leftmost positive root of equation  $\chi_L(z, c) = 0$  and  $m > \lambda$  such that  $\chi_L(m, c) < 0$ , if  $c > c_*$ . Note that for each fix  $\tau$  we have

$$g(s, \tau) \leq C(\tau)s, \quad s \in \mathbb{R}, \quad (4.1)$$

where  $C(\tau)$  is defined in (2.9).

We first consider  $c > c_*$  and for some  $\delta > 0$  we define the functions

$$\phi^-(t) := \delta e^{\lambda t} (1 - e^{(m-\lambda)t}) \chi_{\mathbb{R}_-}(t)$$

and

$$\phi^+(t) := \delta e^{\lambda t}, \quad t \in \mathbb{R}.$$

In addition, we will consider the following space

$$X := \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) : \|\phi\| = \sup_{s \leq 0} e^{-\lambda s/2} |\phi(s)| + \sup_{s \geq 0} e^{-ms} |\phi(s)| < +\infty \right\},$$

and the operator  $\mathcal{A} : \mathcal{R} \rightarrow X$ , where  $\mathcal{R} := \{\phi \in X : \phi^-(t) \leq \phi(t) \leq \phi^+(t), t \in \mathbb{R}\}$  and

$$\mathcal{A}\phi(t) := \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) g(\phi(t-s), \tau) ds.$$

Note that  $\mathcal{R} \subseteq X$  is closed, bounded and convex set of  $X$ . We want to prove the existence of  $\phi \in \mathcal{R}$  such that  $\mathcal{A}\phi = \phi$  and  $\sup_{s \in \mathbb{R}} \phi(s) < +\infty$ . For this, we will prove, in the following Lemma, that  $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{A}$  and  $\mathcal{A}$  is completely continuous map, if we suppose the additional condition

**L:**  $g(s) = Ls$  and  $f_\beta(s) = \left(\beta - \inf_{s \geq 0} f'(s)\right)s$  for all  $s \in [0, \delta]$ .

**Lemma 4.1.** *Assume that  $H_0$ - $H_2$ , the condition (4.1) and **L** hold. Then  $\mathcal{A} : \mathcal{R} \rightarrow \mathcal{R}$  is completely continuous map. Furthermore,  $\mathcal{A}$  has a fix point  $\phi$  in  $\mathcal{R}$  such that  $\sup_{t \in \mathbb{R}} \phi(t) < \infty$ .*

*Proof.* We start with the observation that the proof of this lemma is obtained by similar argument developed in [9, Lemma 13]. Here we only give the main ideas. In fact, we first define the operator

$$L\phi(t) := \int_X C(\tau) d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) \phi(t-s) ds.$$

We now will prove that  $L\phi^+(t) = \phi^+(t)$  and  $\phi^-(t) < L\phi^-(t)$  for all  $t \in \mathbb{R}$ . For this purpose, since  $\chi_L(\lambda) = 0$  we see at once that

$$\begin{aligned} L\phi^+(t) &= \delta e^{\lambda t} \int_X C(\tau) d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) e^{-\lambda s} ds \\ &= \delta e^{\lambda t} (1 - \chi_L(\lambda)) = \delta e^{\lambda t} = \phi^+(t), \end{aligned}$$

by (2.10). Similarly, since  $\chi_L(m) > 0$  we also get that

$$\begin{aligned} L\phi^-(t) &= \delta e^{\lambda t} (1 - e^{(m-\lambda)t}) + \delta e^{\lambda t} \chi_L(m) \\ &> \delta e^{\lambda t} (1 - e^{(m-\lambda)t}) = \phi^-(t). \end{aligned}$$

An analysis similar to that in the proof of [9, Lemma 13], with  $g'(0, \tau)$  and the root  $\lambda$  of  $\chi(z)$  replaced by  $C(\tau)$  and root  $\lambda$  of  $\chi_L(z)$ , respectively, shows that  $\mathcal{A}(\mathcal{R}) \subseteq \mathcal{R}$  and  $\mathcal{A}$  is completely continuous map. In fact, for  $\phi \in \mathcal{R}$

$$\mathcal{A}\phi(t) \leq \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) C(\tau) \phi(t-s) ds = L\phi(t) \leq L\phi^+(t) = \phi^+(t).$$

If there exists  $t_0$  such that  $0 < \phi^-(t_0) \leq \phi(t_0)$ , we would have  $t_0 \in \mathbb{R}_-$  and  $\phi(t_0) \leq \delta$ , and hence,  $g(\phi(t_0), \tau) = C(\tau)\phi(t_0) \geq C(\tau)\phi^-(t_0)$ . In the case that  $\phi^-(t_0) = 0$ , there would be  $g(\phi(t_0), \tau) \geq C(\tau)\phi^-(t_0) = 0$ . Hence

$$\mathcal{A}\phi(t) \geq \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) C(\tau) \phi^-(t-s) ds = L\phi(t) \geq L\phi^-(t) > \phi^-(t).$$

This clearly forces

$$\phi^-(t) \leq \mathcal{A}\phi(t) \leq \phi^+(t), t \in \mathbb{R}.$$

In addition, we observe that  $\mathcal{A}(\mathcal{R})$  is a pre-compact subset of  $\mathcal{R}$ , which follows from the convergence in  $\mathcal{R}$  is uniform on compact subsets of  $\mathbb{R}$ . Furthermore, the functions of  $\mathcal{A}(\mathcal{R})$  are uniformly bounded on every compact subset of  $\mathbb{R}$  and as we have the following estimation

$$\begin{aligned} |\mathcal{A}\phi(t+h) - \mathcal{A}\phi(t)| &\leq \int_X C(\tau) d\rho(\tau) \int_{\mathbb{R}} |N(t+h-u, \tau) - N(t-u, \tau)| \phi(u) du \\ &\leq \int_X C(\tau) d\rho(\tau) \int_{\mathbb{R}} |N(t+s, \tau) - N(s, \tau)| \phi(t-s) ds \\ &\leq \delta e^{k\lambda} \int_X C(\tau) d\rho(\tau) \int_{\mathbb{R}} |N(s+h, \tau) - N(s, \tau)| e^{-\lambda s} ds, \end{aligned}$$

for all  $t \in [-k, k]$  and  $\phi \in \mathcal{R}$ , they are also equicontinuous, by  $|\mathcal{A}\phi(t+h) - \mathcal{A}\phi(t)| \rightarrow 0$ , as  $h \rightarrow 0$  uniformly on every  $[-k, k]$ . In this way, the compactness property of  $\mathcal{A}$  and the dominated convergence theorem imply the continuity of  $\mathcal{A}$ . Finally, the Shauder's fixed point theorem implies that  $\mathcal{A}$  has at least one fixed point  $\phi \in \mathcal{R}$ . Since  $\phi(t) \leq \delta e^{\lambda t}$  and  $\phi$  is a positive solution of (2.1), from Lemma 3.3 we have  $\sup_{t \in \mathbb{R}} \phi(t) < \infty$ , which proves the lemma.  $\square$

**Theorem 4.2.** (Existence of semi-wavefronts) *Let assumptions  $H_1$ - $H_2$  hold. Suppose further conditions (1.2) and (3.1) hold. Then the equation (2.1) has at least one semi-wavefront  $u(x, t) = \phi(x + ct)$  propagating with speed  $c \geq c_*$  such that  $\phi(-\infty) = 0$  and  $\liminf_{t \rightarrow +\infty} \phi(t) > 0$ .*

*Proof.* We define the continuous functions

$$g_n(s) = \begin{cases} Ls, & s \in [0, 1/n], \\ \max \left\{ \frac{L}{n}, g(s) \right\}, & s \geq 1/n, \end{cases} \quad (4.2)$$

and

$$f_{\beta_n}(s) = \begin{cases} \left( \beta - \inf_{s \geq 0} f'(s) \right) s, & s \in [0, 1/n], \\ \max \left\{ \frac{\beta - \inf_{s \geq 0} f'(s)}{n}, f_{\beta}(s) \right\}, & s \geq 1/n. \end{cases} \quad (4.3)$$

Note that  $g_n$  and  $f_{\beta_n}$  satisfy the hypothesis **L** with  $\delta = \frac{1}{n}$ . Moreover, observe that  $g_n$  and  $f_{\beta_n}$  converge uniformly to  $g$  and  $f_{\beta}$  on  $\mathbb{R}_+$ , respectively. If we now define

$$g_n(s, \tau) = \begin{cases} g_n(s), & \tau = \tau_1, \\ f_{\beta_n}(s), & \tau = \tau_2, \end{cases} \quad (4.4)$$

then  $g_n(s, \tau)$  converge uniformly to  $g(s, \tau)$  on  $\mathbb{R}_+$  for  $\tau \in \{\tau_1, \tau_2\}$  and  $g_n$  satisfy the condition  $g_n(s, \tau) \leq C(\tau)s$ ,  $s \in \mathbb{R}$ .

We now consider the operators  $\mathcal{A}_n : \mathcal{R} \rightarrow X$  defined by

$$\mathcal{A}_n\phi(t) := \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) g_n(\phi(t-s), \tau) ds.$$

By Lemma 4.1 we have for each large  $n$  the existence of a positive continuous function  $\phi_n$  such that  $\phi_n(-\infty) = 0$  and

$$\mathcal{A}_n\phi_n(t) = \phi_n(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) g_n(\phi_n(t-s), \tau) ds.$$

Consequently, we have shown that the functions  $\phi_n$  is a positive solution of equation

$$y''(t) - cy'(t) - \beta y(t) + f_{\beta_n}(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s, w) g_n(y(t - cs - w)) dw ds = 0, \quad (4.5)$$

with speed  $c > c_*$ .

Proceeding analogously to the proof of Lemma 3.3, we can obtain that  $\phi_n$  are bounded functions such that

$$\phi_n(t) \leq \frac{\max\{L, \sup_{u \geq 0} g(u)\}}{\inf_{t \geq 0} f'(t)} < \infty.$$

Moreover, for all sufficiently large  $n$ , and with the same  $\xi_1$  and  $\xi_2$  given in Lemma 3.6 the properties of Lemma 3.4 hold. In consequence, we have

$$\liminf_{t \rightarrow +\infty} \phi_n(t) > \xi_1.$$

Consequently, with a similar argument to [9, Corollary 16], we can prove that  $\{\phi_n\}$  are equicontinuous on  $\mathbb{R}$ . Thus there exists a subsequence  $\phi_{n_j}$  which converges uniformly on compact sets to some bounded  $\phi \in C(\mathbb{R}, \mathbb{R})$ . By Lebesgue's dominated convergence theorem,  $\phi$  satisfies the equation 2.6 and

$$\phi(-\infty) = 0, \quad \liminf_{t \rightarrow +\infty} \phi(t) \geq \xi_1 > 0.$$

Finally, for the case  $c = c_*$ , we define  $c_n := \frac{(n+1)c_*}{n}$ . Since  $c_n > c_*$ , the pervious result assures the existence of positive bounded solutions  $\psi_n$  to (2.1) such that  $\psi_n(-\infty) = 0$ ,  $\liminf_{t \rightarrow +\infty} \psi_n(t) \geq \xi_1$  and

$$\begin{aligned} \psi_n(t) &= \frac{1}{\sigma(c_n)} \left( \int_{-\infty}^t e^{\nu(c_n)(t-s)} (\mathcal{G}\psi_n)(s) ds + \int_t^{+\infty} e^{\mu(c_n)(t-s)} (\mathcal{G}\psi_n)(s) ds \right) \\ &= \int_{\mathbb{R}} k_1(t-s) (\mathcal{G}\psi_n)(s) ds, \quad t \in \mathbb{R}, \end{aligned} \quad (4.6)$$

where

$$k_1(s) = (\sigma(c_n))^{-1} \begin{cases} e^{\nu(c_n)s}, & s \geq 0 \\ e^{\mu(c_n)s}, & s < 0 \end{cases},$$

$\sigma(c_n) = \sqrt{c_n^2 + 4\beta}$ ,  $\nu(c_n) < 0 < \mu(c_n)$  are the roots of  $z^2 - c_n z - \beta = 0$  and the operator  $\mathcal{G}$  is defined as

$$(\mathcal{G}\psi)(t) := \int_0^\infty \int_{\mathbb{R}} K(s, w) g(\psi(t - c_n s - w)) dw ds + f_\beta(\psi(t)),$$

Now, differentiating (4.6) we find that

$$|\psi'_n(t)| \leq \frac{1}{\sigma(c_*)} \left( \sup_{u \geq 0} g(u) + \frac{\sup_{u \geq 0} g(u)}{\inf_{s \geq 0} f'(s)} \right).$$

In consequence,  $\{\psi_n\}$  is pre-compact in the compact open topology of  $C(\mathbb{R}, \mathbb{R})$  and we find a subsequence  $\{\psi_{n_j}\}$  which converges uniformly on compacts to some bounded function  $\psi \in C(\mathbb{R}, \mathbb{R})$ . In addition, Lebesgue's dominated convergence theorem implies that  $\psi$  is a solution of (2.1) with  $c = c_*$  such that  $\psi(-\infty) = 0$  and  $\liminf_{t \rightarrow +\infty} \psi(t) \geq \xi_1$ .  $\square$

*Remark 4.3.* Note that if we assume that  $L = g'(0)$  and  $f'(s) \geq f'(0), t \geq 0$ , then  $c_* = c_*$ . In [1] we have proved that for any  $c < c_*$  the equation (2.1) has no semi-wavefront solution propagating with speed  $c$  vanishing at  $-\infty$ .

**Corollary 4.4.** (*Existence of wavefronts*) Assume all conditions of Theorem 4.2 are fulfilled. If equation  $f(s) = g(s)$  has only two solutions: 0 and  $\kappa$ , with  $\kappa$  being globally attracting with respect to  $f^{-1} \circ g$ , then the equation (2.1) has at least one wavefront  $u(x, t) = \phi(x + ct)$  propagating with speed  $c \geq c_*$  such that  $\phi(+\infty) = \kappa$ .

*Remark 4.5.* Sufficient conditions to ensure the global stability of  $f^{-1} \circ g$  are given in [20].

## 5. Applications.

In this section, we apply Theorem 1.1 to some non-local reaction-diffusion epidemic and population models with distributed time delay, studied in [3, 6, 10, 17, 19, 21, 23, 24, 26].

**An application to the epidemic dynamics:** Consider the following reaction-diffusion model with distributed delay

$$\begin{cases} u_t(t, x) = du_{xx}(t, x) - f(u(t, x)) + \int_{\mathbb{R}} K(x - y)v(t, y)dy \\ v_t(t, x) = -\alpha v(t, x) + \int_0^\infty g(u(t - s, x))P(ds), \end{cases} \quad (5.1)$$

where  $\alpha, d > 0$ ,  $x \in \mathbb{R}, t \geq 0$ , and  $P$  is a probability measure on  $\mathbb{R}_+$ . The functions  $u(t, x)$  and  $v(t, x)$  denote the densities of the infectious agent and the infective human population at a point  $x$  in the habitat at time  $t$ , respectively (see [19, 23, 24, 26]). Note that system (5.1) can be seen as a generalization of the systems studied in the cited works. However, here the nonnegative kernel  $K$  can be asymmetric and normalized by  $\int_{\mathbb{R}} K(w)dw = 1$ , and the function  $g$  can be non-monotone. By scaling the variables, we can suppose that  $d = 1$ .

Now, suppose that  $(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct))$  is a semi-wavefront solution of system (5.1) with speed  $c$ , i.e. the continuous non-constant uniformly bounded functions  $u(t, x) = \phi(x + ct)$  and  $v(t, x) = \psi(x + ct)$  are positives and satisfy the condition  $\phi(-\infty) = \psi(-\infty) = 0$ . Then the wave profiles  $\phi$  and  $\psi$  must satisfy the following system:

$$\begin{cases} \phi''(t) - c\phi'(t) - f(\phi(t)) + \int_{\mathbb{R}} K(u)\psi(t - u)du = 0 \\ c\psi'(t) + \alpha\psi(t) - \int_0^\infty g(\phi(t - cs))P(ds) = 0. \end{cases} \quad (5.2)$$

As it was obtained in [1],  $\psi$  satisfies

$$\psi(t) = \int_0^\infty g(\phi(t - cw))K_2(w)dw, \quad c \neq 0, \quad (5.3)$$

where

$$K_2(w) = \int_0^w e^{-\alpha(w-r)}P(dr),$$

and if  $c = 0$ , then  $\alpha\psi(t) = g(\phi(t))$ . In consequence,  $\phi(t)$  should satisfy the integral equation

$$\begin{aligned}\phi(t) &= \frac{1}{\sigma(c)} \left( \int_{-\infty}^t e^{\nu(c)(t-s)} (\mathcal{G}\phi)(s) ds + \int_t^{+\infty} e^{\mu(c)(t-s)} (\mathcal{G}\phi)(s) ds \right) \\ &= \int_{\mathbb{R}} k_1(t-s) (\mathcal{G}\phi)(s) ds,\end{aligned}\tag{5.4}$$

where

$$k_1(s) = (\sigma(c))^{-1} \begin{cases} e^{\nu(c)s}, & s \geq 0 \\ e^{\mu(c)s}, & s < 0 \end{cases},$$

$\sigma(c) = \sqrt{c^2 + 4\beta}$ ,  $\nu(c) < 0 < \mu(c)$  are the roots of  $z^2 - cz - \beta = 0$  and the operator  $\mathcal{G}$  is defined as

$$(\mathcal{G}\phi)(t) := \int_{\mathbb{R}} K(u) \psi(t-u) du + f_{\beta}(\phi(t)), \quad f_{\beta}(s) = \beta s - f(s), \beta > f'(0).$$

Thus, the profile  $\phi$  also must satisfy the equation

$$\phi(t) = (k_1 * k_2) * g(\phi)(t) + k_1 * f_{\beta}(\phi)(t).\tag{5.5}$$

A similar argument can be applied when  $c = 0$ .

Conversely, if  $\phi$  is a semi-wavefront solution of (5.4) propagating with speed  $c$ , then the function  $\psi$  defined by (5.4) is well defined and  $\psi(-\infty) = 0$ , by the Lebesgue's dominated convergence theorem. Moreover, we get that

$$|\psi(t)| \leq \frac{\sup_{u \geq 0} g(u)}{\alpha}, t \in \mathbb{R}.$$

Since the process developed in [1] to obtain (5.3) and (5.4) is invertible, it follows that  $(\phi(t), \psi(t))$  is a semi-wavefront solution to (5.2) propagating with speed  $c$ . In consequence we have the following theorem.

**Lemma 5.1.** *The following affirmations are true.*

1. If  $(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct))$  is a semi-wavefront of (5.1) with speed  $c$ , then  $(\phi(t), \psi(t))$  is a semi-wavefront of (5.2) with speed  $c$ .
2. Let  $\phi(t)$  be a semi-wavefront of (5.5) with speed  $c$ . If  $\psi(t)$  is defined by (5.3), then  $(\phi(t), \psi(t))$  is a semi-wavefront of (5.2) with speed  $c$  and  $(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct))$  is also a semi-wavefront of (5.1).

Next, the characteristic function  $\chi$  becomes:

$$\chi(z) = -\frac{z^2 - cz - f'(0) + \frac{g'(0)}{cz + \alpha} \int_0^{\infty} e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw}{\beta + cz - z^2}$$

when  $cz + \alpha > 0$ . Consequently, from [1], we obtain

$$\chi_0(z, c) = z^2 - cz - f'(0) + \frac{g'(0)}{cz + \alpha} \int_0^{\infty} e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw,$$

and

$$\chi_L(z, c) = z^2 - cz - \inf_{s \geq 0} f'(s) + \frac{L}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw.$$

In this way, let  $c_*$  and  $c_*$  be the minimal value of  $c$  for which  $\chi_0(z, c) = 0$  and  $\chi_L(z, c) = 0$  have at least one positive root, respectively. Then we can now formulate the following result:

**Theorem 5.2.** *Let assumptions  $H_0$ - $H_2$  hold. If  $g(s) \leq Ls$  and  $f(s) \geq \inf_{s \geq 0} f'(s)s$ ,  $s > 0$ , then the system (5.1) admits at least one semi-wavefront solution*

$$(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)), \quad \phi(-\infty) = \psi(-\infty) = 0,$$

for each admissible wave speed  $c \geq c_*$ . If  $\inf_{s \geq 0} f'(s) = f'(0)$ , then the existence holds for each  $c \geq c_*$ . Furthermore, the system (5.1) has no semi-wavefront solution propagating with speed  $c < c_*$ .

*Remark 5.3.* Theorem 5.2 completes or improves some results of [19, 23, 24, 26]. In fact, in these references the monotone case was studied, except [24]. It should be noted that in [23, 26], isotropic kernels were considered.

**An application to the population dynamics:** Let  $u$  and  $v$  denote the numbers of mature and immature population of a single species at time  $t \geq 0$ , respectively. We will study the system

$$\begin{cases} u_t(t, x) = du_{xx}(t, x) - f(u(t, x)) + \int_0^\infty \int_{\mathbb{R}} K(s, w)g(u(t-s, x-w))dw ds \\ v_t(t, x) = Dv_{xx}(t, x) - \gamma v(t, x) + g(u(t, x)) - \int_0^\infty \int_{\mathbb{R}} K(s, w)g(u(t-s, x-w))dw ds, \end{cases} \quad (5.6)$$

where  $\gamma, D, d > 0$  and the nonnegative kernel  $K$  can be asymmetric. Note that by scaling the variables, we can suppose that  $d = 1$ . Now, observe that in the system (5.6) the first equation can be solved independently of the second. In this way, if the system (5.6) admits a semi-wavefront solution

$$(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)), \quad \phi(-\infty) = \psi(-\infty) = 0,$$

with speed  $c$ , then  $v(t, x) = \psi(x + ct)$  must satisfy the immature equation

$$D\psi''(t) - c\psi'(t) - \gamma\psi(t) + (\mathcal{H}\phi)(t) = 0,$$

where the operator  $\mathcal{H}$  is defined by

$$(\mathcal{H}\phi)(t) = g(\phi(t)) - \int_0^\infty \int_{\mathbb{R}} K(s, w)g(\phi(t - cs - w))dw ds.$$

If  $\phi$  is bounded, we get that  $\psi$  can be represented by

$$\psi(t) = \int_{\mathbb{R}} k_1(t-s)(\mathcal{H}\phi)(s)ds = \int_{\mathbb{R}} k_1(s)(\mathcal{H}\phi)(t-s)ds,$$

where

$$k_1(s) = \left( \sqrt{c^2 + 4D\gamma} \right)^{-1} \begin{cases} e^{\tilde{\nu}(c)s}, & s \geq 0 \\ e^{\tilde{\mu}(c)s}, & s < 0 \end{cases}$$

and  $\tilde{\nu}(c) < 0 < \tilde{\mu}(c)$  are the roots of  $Dz^2 - cz - \gamma = 0$ . In addition,  $\mathcal{H} \in C(\mathbb{R}_+, \mathbb{R}_+)$  and  $\mathcal{H}(0) = 0$ . Now, if  $\phi(-\infty) = 0$ , then we have  $\mathcal{H}(\phi(-\infty)) = 0$ , and in consequence, the Lebesgue's theorem of dominated convergence implies that  $\psi(-\infty) = 0$ . Thus we obtain the following lemma.

**Lemma 5.4.** *The second equation of the system (5.6) has a semi-wavefront  $v(t, x) = \psi(x + ct)$  with  $\psi(-\infty) = 0$  when  $u(t, x) = \phi(x + ct)$  is a semi-wavefront of the first equation of the system (5.6).*

Finally, consider the characteristic functions  $\chi_0(z, c)$  and  $\chi_L(z, c)$  associated with the mature equation of system (5.6) and  $c_*, c_*$  defined in Section 1. Then the following theorem is a direct consequence of Theorem 1.1.

**Theorem 5.5.** *Let assumptions  $H_1$ - $H_2$  hold. If  $g(s) \leq Ls$  and  $f(s) \geq \inf_{s \geq 0} f'(s)s$ ,  $s > 0$ , then the system (5.6) admits at least one semi-wavefront solution*

$$(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)), \quad \phi(-\infty) = \psi(-\infty) = 0,$$

for each admissible wave speed  $c \geq c_*$ . If  $\inf_{s \geq 0} f'(s) = f'(0)$ , then the existence holds for each  $c \geq c_*$ . Furthermore, the system (5.6) has no semi-wavefront solution propagating with speed  $c < c_*$ .

*Remark 5.6.* We note that Theorem 5.5 completes or improves some results of [6, 10, 19, 21], where the non-existence or the uniqueness was established under assumptions that  $K$  is Gaussian or symmetric kernel, and  $g$  monotone. In [10, 21] only the particular cases  $f(s) = \beta s^2$  and  $g(s) = s$ , were studied, and in [19], the assumptions were either  $f(s) = f'(0)$  or  $g(s) = g'(0)s$ .

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## References

- [1] M. Aguerrea, On uniqueness of semiwavefronts for non-local delayed reaction-diffusion equations, J. Mathematical Analysis and Applications. 422 (2015) 1007-1025
- [2] M. Aguerrea, Existence of fast positive wavefronts for a non-local delayed reaction-diffusion equation, Nonlinear Analysis. 72 (2010) 2753-2766.
- [3] W.G. Aiello, H.I. Freedman, A time-delay model of single species growth with stage structure, Math. Biosci. 101 (1990) 139-153.
- [4] N. F. Britton, Spatial structures and periodic traveling waves in an integro- differential reaction-diffusion population model, SIAM J. Appl. Math. 50 (1990) 1663-1688.
- [5] J. Fang, X. Zhao, Existence and uniqueness of traveling waves for non-monotone integral equations with applications, J. Differ. Equations. 248 (2010) 2199-2226.
- [6] J. Fang, J. Wei, X. Zhao, Spatial dynamics of a nonlocal and time-delayed reaction-diffusion system, Journal of Differ. Equations. 245 (2008) 2749-2770.
- [7] T. Faria, S. Trofimchuk, Nonmonotone travelling waves in a single species reaction-diffusion equation with delay, J. Differential Equations. 228 (2006) 357-376.



- [8] T. Faria, W. Huang, J. Wu, Traveling waves for delayed reaction-diffusion equations with non-local response, *Proc. Roy. Soc. London Sect. A.* 462 (2006) 229-261.
- [9] C. Gomez, H. Prado, S. Trofimchuk, Separation dichotomy and wavefronts for a nonlinear convolution equation, *J. Mathematical Analysis and Applications.* 420(2014) 1-19.
- [10] S. A. Gourley, Y. Kuang, Wavefronts and global stability in time-delayed population model with stage structure, *Proc. R. Soc. A.* 459 (2003) 1563-1579.
- [11] S. A. Gourley, J. So, Extinction and wavefront propagation in a reaction-diffusion model of a structured population with distributed maturation delay, *Proc. Royal Soc. of Edinburgh.* 133A (2003) 527-548.
- [12] S. A. Gourley, J. So, J. Wu, Non-locality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics. *J. Math. Sciences.* 124 (2004) 5119-5153.
- [13] W. T. Li, S. Ruan, Z.C. Wang, On the diffusive Nicholson's blowies equation with nonlocal delay, *J. Nonlinear Science.* 17 (2007) 505-525.
- [14] D. Liang, J. Wu, Travelling waves and numerical approximations in a reaction-advection-diffusion equation with non-local delayed effects, *J. Nonlinear Science.* 13 (2003) 289-310.
- [15] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, *J. Differential Equations.* 237 (2007) 259-277.
- [16] M. Mei, J. So, Stability of strong traveling waves for a non-local time-delayed reaction-diffusion equation, *Proc. Roy. Soc. Edinburgh.* A 138 (2008) 551-568.
- [17] J. Al-Omari, S.A. Gourley, Monotone wave-fronts in a structured population model with distributed maturation delay, *IMA J. Appl. Math.* 70 (2005) 858-879.
- [18] J. So, J. Wu, X. Zou, A reaction-diffusion model for a single species with age structure. I, Travelling wave fronts on unbounded domains, *Proc. Roy. Soc.* 457 (2001) 1841-1853.
- [19] H. R. Thieme, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models, *J. Differential Equations.* 195 (2003) 430-470.
- [20] E. Trofimchuk, P. Alvarado, S. Trofimchuk, On the geometry of wave solutions of a delayed reaction-diffusion equation, *J. Differential Equations.* 246 (2009) 1422-1444.
- [21] Z.-C. Wang, W.T. Li, S. Ruan, Traveling Fronts in Monostable Equations with Nonlocal Delayed Effects, *J. Dyn. Diff. Equat.* 20 (2008) 573-607.
- [22] H. Wang, On the existence of traveling waves for delayed reaction-diffusion equations, *J. Differential Equations* 247. (2009) 887-905.
- [23] S. Wu, S. Liu, Asymptotic speed of spread and traveling fronts for a nonlocal reaction-diffusion model with distributed delay, *Applied Mathematical Modelling.* 33 (2009) 2757- 2765.
- [24] S. Wu, S. Liu, Existence and uniqueness of traveling waves for non-monotone integral equations with application *Journal of Mathematical Analysis and Applications.* 365 (2010) 729-741.
- [25] Z. Xu, P. Weng, Traveling waves for nonlocal and non-monotone delayed reaction-diffusion equations, *Acta Mathematica Sinica, English Series* <http://maths.scnu.edu.cn/Uploadfiles/201352214355539.pdf>.
- [26] D. Xu, X. Zhao, Asymptotic speed of spread and traveling wave for nonlocal epidemic model, *Discrete and Continuous Dynamical Systems-Series B.* 5 (2005) Number 4.
- [27] T. Yi, Y. Chen, J. Wu, Unimodal dynamical systems: Comparison principles, spreading speeds and traveling waves, *J. Differential Equations.* 254 (2013) 3538-3572.